Spin correlations and consequences of quantum–mechanical coherence

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Abstract

The difference in the properties of the spin correlation tensor for factorizable and nonfactorizable two-particle states is analyzed. The inequalities for linear combinations of the components of this tensor are obtained for the case of incoherent mixtures of factorizable two-particle spin states. They include the well known Bell inequalities and can be violated for coherent superpositions of two–particle spin states. The possibility to verify the consequences of the quantum-mechanical coherence is discussed using the angular correlations in the asymmetric (parity violating) decays of the pairs of spin-1/2 particles (muons, top-quarks or Λ-hyperons), the coherence arising either from the production dynamics or due to the effect of quantum statistics.

1 Introduction

It is well known that correlations in the detection of nonfactorizable two-particle states represent a manifestation of the quantum-mechanical effect first considered by Einstein, Podolsky and Rosen [1]. The essence of this effect is as follows. If the two-particle state is not factorizable, the character of the measurements performed on the first particle determines the readout of the detector analyzing the state of the second particle, even if the detectors were situated at a large distance. To demonstrate this effect, consider a nonfactorizable two-particle state as a coherent superposition of pairs of one-particle states:

$$\mid \Phi\rangle^{(1,2)} = \sum_i \sum_k c_{ik} \mid i\rangle^{(1)} \mid k\rangle^{(2)}, \tag{1}$$

where c_{ik} are complex numbers normalized to unity, $\sum_i \sum_k \mid c_{ik} \mid^2 = 1$. In this case, the amplitude to observe the two-particle state (1) by two one-particle detectors selecting the states $\mid L\rangle^{(1)}$ and $\mid M\rangle^{(2)}$ results from the interference of pairs of one-particle states: $A_{LM} = \sum_i \sum_k c_{ik} \langle L \mid i\rangle^{(1)} \langle M \mid k\rangle^{(2)}$. Clearly, the selection of different states $\mid L\rangle^{(1)}$ and $\mid M\rangle^{(1)}$ for the first particle then leads to the different states of the second particle:

$$\mid \Psi\rangle_L^{(2)} = \sum_i \sum_k c_{ik} \langle L \mid i\rangle \mid k\rangle^{(2)}, \qquad \mid \Psi\rangle_M^{(2)} = \sum_i \sum_k c_{ik} \langle M \mid i\rangle \mid k\rangle^{(2)}. \tag{2}$$

Let us note that the states $\mid \Psi\rangle_L^{(2)}$ and $\mid \Psi\rangle_M^{(2)}$ can be the eigenfunctions of noncommuting operators. As a result, in the presence of the correlations, the one-particle state is not a pure one - it should be described by the density matrix and not by the wave function.

The fact that the state of one of two particles can be managed without a direct force action on it, Einstein considered as a paradox pointing to the incompleteness of the quantum-mechanical description [1]. However, it has become clear that here we deal with the correlation effect connected with coherent properties of quantum-mechanical superpositions. The properties of $K^0\overline{K}^0$-pairs provide an impressive example: the detection of one of two neutral kaons through its decay or its interaction determines the internal state of the second kaon [2-5].

The polarization correlations, discussed in present paper, belong to the same group of phenomena [6-8]. It should be emphasized that namely in these cases the so-called Bell inequalities are violated. These inequalities were derived at the probability level without taking into account the coherent properties of the quantum-mechanical superpositions [9, 10]. We prove here a class of the inequalities, including those of Bell, based on the assumption of the factorizability of the two-particle density matrix, i.e. on its reduction to a sum of the direct products of one–particle density matrices with the nonnegative coefficients. Clearly, such a form of the density matrix corresponds to a classical probabilistic description and cannot account for the coherent quantum–mechanical effects. The violation of the Bell–type inequalities thus clearly manifests the coherent nature of quantum mechanics. We show here that the correlations in the parity-violating decays of pairs of spin-1/2 particles can serve as a sensitive and relatively simple test of this coherence, extending the tests already done with optical photons and secondary scatterings of low energy protons (see, e.g., a review [10]).

2 Two-particle density matrix and spin correlations

For two spin-1/2 particles, the normalized spin density matrix, with the sum of the diagonal elements ("trace") $\mathrm{tr}_{(1,2)}\hat{\rho}^{(1,2)} = 1$, has the following general structure (see, e.g., [7])

$$\hat{\rho}^{(1,2)} = \frac{1}{4}[\hat{I}^{(1)} \otimes \hat{I}^{(2)} + (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{P}_1) \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{P}_2) + \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik}\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}]. \quad (3)$$

Here $\hat{I}$ is the two-row unit matrix, $\hat{\boldsymbol{\sigma}}$ is the Pauli vector operator, $T_{ik} = \langle\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}\rangle$ are the components of the correlation tensor. The corresponding one-particle density matrices contain the polarization vectors $\mathbf{P}_l = \langle\hat{\boldsymbol{\sigma}}^{(l)}\rangle$ only: $\hat{\rho}^{(l)} = \frac{1}{2}(\hat{I} + \hat{\boldsymbol{\sigma}}\vec{P}_l),\ l = 1, 2$. In the absence of correlations the factorization takes place:

$$T_{ik} = P_{1i}P_{2k}, \qquad \hat{\rho}^{(1,2)} = \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}. \quad (4)$$

Let two analyzers select the states of the first and the second particle with the polarization vectors $\boldsymbol{\zeta}^{(1)}$ and $\boldsymbol{\zeta}^{(2)}$. Then the detection probability W depends linearly on the polarization parameters of the two-particle system as well as on the final polarization parameters fixed by detectors, and it can be obtained by the substitution of the matrices $\hat{\sigma}_i^{(1)}$ and $\hat{\sigma}_k^{(2)}$ in the expression (6) with the corresponding projections $\zeta_i^{(1)}$ and $\zeta_k^{(2)}$. As a result [7]

$$W \sim 1 + \mathbf{P}_1\boldsymbol{\zeta}^{(1)} + \mathbf{P}_2\boldsymbol{\zeta}^{(2)} + \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik}\zeta_i^{(1)}\zeta_k^{(2)}. \quad (5)$$

Let only the polarization vector $\boldsymbol{\zeta}^{(1)}$ of the first particle is measured. Then, due to the correlations, the spin state of the second particle is described by the normalized density matrix

$$\hat{\tilde{\rho}}^{(2)} = \frac{1}{2}(1+\boldsymbol{\zeta}^{(1)}\mathbf{P}_1)^{-1}[(1+\boldsymbol{\zeta}^{(1)}\mathbf{P}_1)\hat{I} + \hat{\boldsymbol{\sigma}}\mathbf{P}_2 + \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik}\zeta_i^{(1)}\hat{\sigma}_k]. \tag{6}$$

In this case the polarization vector of the second particle has the components

$$\tilde{\zeta}_k^{(2)} = (P_{2k} + \sum_{i=1}^{3} T_{ik}\zeta_i^{(1)})/(1+\boldsymbol{\zeta}^{(1)}\mathbf{P}_1). \tag{7}$$

In the case of independent particles, when the factorization takes place (see Eq. (4), the detection of the spin state of the first particle does not influence the polarization of the second particle: $\tilde{\boldsymbol{\zeta}}^{(2)} = \mathbf{P}_2$.

The situation is of interest when both one-particle states are unpolarized and the polarization vectors $\mathbf{P}_1$ and $\mathbf{P}_2$ vanish. Then, in accordance with Eq. (7), the spin effects are completely determined by the correlation tensor T_{ik}: $\tilde{\zeta}_k^{(2)} = \sum_{i=1}^{3} T_{ik}\zeta_i^{(1)}$. If the one-particle states are unpolarized and the spin correlations are absent, then $\tilde{\boldsymbol{\zeta}}^{(2)} = 0$ for any selection of the vector $\boldsymbol{\zeta}^{(1)}$.

It may be useful to calculate the polarization vectors $\mathbf{P}_l = \langle\Psi_{SM}|\hat{\boldsymbol{\sigma}}^{(l)}|\Psi_{SM}\rangle$ and the correlation tensor $T_{ik} = \langle\Psi_{SM}|\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}|\Psi_{SM}\rangle$ in the pure singlet ($S=0$) and triplet ($S=1$) states $|\Psi_{SM}\rangle$.

The singlet state of two spin-1/2 particles is a typical example of a nonfactorizable two-particle state. It is described by the spin wave function

$$|\,\Psi\rangle_{00} = \frac{1}{\sqrt{2}}(|+1/2\rangle_z^{(1)}\,|-1/2\rangle_z^{(2)} - |-1/2\rangle_z^{(1)}\,|+1/2\rangle_z^{(2)}), \tag{8}$$

corresponding to rigidly correlated particle spins with the spin projections opposite for any choice of the quantization axis z; at the same time, the particle polarizations are equal to zero:

$$\mathbf{P}_1 = \mathbf{P}_2 = 0, \qquad T_{ik} = -\delta_{ik}. \tag{9}$$

Consider further the triplet states which are polarized and aligned along the spin axis unit vector $\mathbf{e}$, so that the triplet density matrix is diagonal in the representation of the spin projections onto the axis $\mathbf{e}$: $\rho_{mm'} = w_m\delta_{mm'}$. The states with the spin projections $M_{\mathbf{e}} = +1, -1, 0$ can be respectively written as

$$\begin{gathered} |\,\Psi\rangle_{1+1} = |+1/2\rangle_{\mathbf{e}}^{(1)}\,|+1/2\rangle_{\mathbf{e}}^{(2)}, \qquad |\,\Psi\rangle_{1-1} = |-1/2\rangle_{\mathbf{e}}^{(1)}\,|-1/2\rangle_{\mathbf{e}}^{(2)}, \\ |\,\Psi\rangle_{10} = \tfrac{1}{\sqrt{2}}(|+1/2\rangle_{\mathbf{e}}^{(1)}\,|-1/2\rangle_{\mathbf{e}}^{(2)} + |-1/2\rangle_{\mathbf{e}}^{(1)}\,|+1/2\rangle_{\mathbf{e}}^{(2)}). \end{gathered} \tag{10}$$

Normalizing the corresponding occupancies to unity: $w_+ + w_- + w_0 = 1$, one has:

$$\mathbf{P}_1 = \mathbf{P}_2 = (w_+ - w_-)\mathbf{e}, \quad T_{ik} = (1-3w_0)e_ie_k + w_0\delta_{ik}. \tag{11}$$

In case of the unpolarized triplet $w_+ = w_- = w_0 = 1/3$, so that $\mathbf{P}_1 = \mathbf{P}_2 = 0,\ T_{ik} = \delta_{ik}/3$.

Note that for a general triplet density matrix $\rho_{mm'}$:

$$P_1 = \sqrt{2}\Re(\rho_{+0} + \rho_{-0}),\ P_2 = -\sqrt{2}\Im(\rho_{+0} - \rho_{-0}),\ P_3 = \rho_{++} - \rho_{--},$$

$$T_{11} = \rho_{00} + 2\Re\rho_{+-},\ T_{22} = \rho_{00} - 2\Re\rho_{+-},\ T_{33} = \rho_{++} + \rho_{--} - \rho_{00} = 1 - 2\rho_{00},$$

$$T_{12} = T_{21} = -2\Im\rho_{+-},\ T_{13} = T_{31} = \sqrt{2}\Re(\rho_{+0} - \rho_{-0}),\ T_{23} = T_{32} = -\sqrt{2}\Im(\rho_{+0} + \rho_{-0}). \tag{12}$$

One may see that the diagonal components of the triplet correlation tensor always satisfy the equalities ($w_0 \equiv \rho_{00}$):

$$T_{11} + T_{22} = 2w_0, \quad T_{33} = 1 - 2w_0, \quad \mathrm{tr}T = 1. \tag{13}$$

3 Analyzers of the spin polarization

As for the polarization analyzers, one can use the secondary scattering events and exploit the fact that the scattering of a particle with spin 1/2 on a spinless or unpolarized target selects the states polarized parallel to the normal of the scattering plane. The final polarization vectors in Eq. (5) are then the analyzing powers: $\boldsymbol{\zeta}^{(l)} = \alpha_l(\mathbf{p}_l, \theta_l)\mathbf{n}^{(l)},\ l = 1, 2$. Here $\mathbf{p}_l$ are the three-momenta of the produced particles, θ_l are the scattering angles, $\mathbf{n}^{(l)}$ are the unit vectors parallel to the scattering plane normals and α_l are the left-right azimuthal asymmetry parameters vanishing at zero scattering angle. According to the Wolfenstein theorem [11], the analyzing power coincides with the polarization vector arising as a result of the elastic scattering of the unpolarized particle on the same target. Inserting the analyzing powers $\boldsymbol{\zeta}^{(l)}$ in Eq. (5), one obtains the probability of the simultaneous detection of two particles, produced in the same collision and subsequently scattered on an unpolarized target, describing the correlation of the scattering planes [7].

It is interesting to note that if two unpolarized particles are produced and only one of them is scattered on an unpolarized target, then the spin correlation results in the polarization of the other (unscattered) particle created together with the scattered one in the same collision event (see Eq. (7)): $\tilde{\zeta}_k^{(2)} = \alpha_1(\mathbf{p}_1, \theta_1)\sum_{i=1}^{3} T_{ik} n_i^{(1)}$. This phenomenon makes it possible, in principle, to prepare particle beams with regulated spin polarization without any direct action on the particles in the polarized beam.

Compared to the secondary scattering, there is often a more easy way to analyze the spin states of produced spin–1/2 particles using their asymmetric (parity violating) decays (see, e.g., [12, 13]). The parity violation is characterized by the asymmetry parameter α measuring the strength of the correlation between the particle polarization $\mathbf{P}$ and the decay analyzer unit vector $\mathbf{n}$.

For example, $\alpha = 0.642$ for the decay $\Lambda \to p\pi^-$ with the decay analyzer chosen parallel to the proton momentum in the Λ-rest frame. Other examples are the muon decay: $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ with the asymmetry parameter $\alpha_e = -1/3$, and the top–quark decay: $t \to bl^+\nu_l\ (b\bar{d}u,\ b\bar{s}c)$ with the asymmetry parameters [14] $\alpha_b \doteq -0.4$, $\alpha_{\nu_l(u,c)} \doteq -0.33$ and $\alpha_{l^+(\bar{d},\bar{s})} = 1$. Due to the CP invariance, the asymmetry parameters for the decays of corresponding antiparticles are the same up to the opposite signs: $\bar{\alpha} = -\alpha$.

Using the fact that the decay selects the spin projections parallel to the decay analyzer $\hat{\mathbf{n}}$, one can obtain the double angular distribution of the decay analyzers by inserting in

Eq. (5) the analyzing powers $\boldsymbol{\zeta}_1 = \alpha_1 \mathbf{n}^{(1)}$ and $\boldsymbol{\zeta}_2 = \alpha_2 \mathbf{n}^{(2)}$ (see also [13]):

$$W(\Omega_1, \Omega_2) = \frac{1}{(4\pi)^2}[1 + \alpha_1 \mathbf{P_1} \mathbf{n}^{(1)} + \alpha_2 \mathbf{P_2} \mathbf{n}^{(2)} + \alpha_1 \alpha_2 \sum_{i=1}^{3} \sum_{k=1}^{3} T_{ik} n_i^{(1)} n_k^{(2)}]. \tag{14}$$

Integrating Eq. (14) over all angles except the angle θ_{12} between the decay analyzers $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$, one gets (see also [12, 13]):

$$W(x) = \frac{1}{2}[1 + \frac{1}{3}\mathrm{tr}T \alpha_1 \alpha_2 x], \tag{15}$$

where $x \equiv \cos\theta_{12} = \mathbf{n}^{(1)}\mathbf{n}^{(2)}$ and the trace of the correlation tensor tr$T$ can be expressed through the rotation invariant combination of the two–particle density matrix elements, such as the singlet or triplet fractions $\rho_s \equiv \rho_0$ or $\rho_t \equiv \rho_1$, $\rho_s + \rho_t = 1$. Using the fact that the eigen values of the operator $\hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)}$ are equal to -3 and 1 for the singlet and triplet states respectively, one has

$$\mathrm{tr}T = \rho_t - 3\rho_s \equiv 4\rho_t - 3. \tag{16}$$

Particularly, in agreement with Eqs. (9) and (12), the correlations between the decay analyzers in the pure singlet and triplet states are of opposite signs and differ in magnitude by a factor of three: $\mathrm{tr}T^s = -3\mathrm{tr}T^t = -3$.

The polarization vectors and the correlation tensor can be easily determined experimentally by the method of moments. For example, using Eq. (14), one can calculate the diagonal components of the correlation tensor according to the expressions:

$$T_{11} = \tfrac{9}{\alpha_1\alpha_2} \left\langle \sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2 \right\rangle; \quad T_{22} = \tfrac{9}{\alpha_1\alpha_2} \left\langle \sin\theta_1 \sin\phi_1 \sin\theta_2 \sin\phi_2 \right\rangle; \\ T_{33} = \tfrac{9}{\alpha_1\alpha_2} \left\langle \cos\theta_1 \cos\theta_2 \right\rangle, \tag{17}$$

where θ_l and ϕ_l denote respectively the polar and azimuthal angles of the decay analyzers $\mathbf{n}^{(l)}$, $l = 1, 2$. One can directly calculate the sums (see also Eq. (15)):

$$T_{11} + T_{22} = \frac{9}{\alpha_1\alpha_2} \left\langle \sin\theta_1 \sin\theta_2 \cos\phi_{12} \right\rangle; \quad \mathrm{tr}T = \frac{9}{\alpha_1\alpha_2} \left\langle \cos\theta_{12} \right\rangle, \tag{18}$$

where $\phi_{12} = \phi_1 - \phi_2$ and $\cos\theta_{12} = \mathbf{n}^{(1)}\mathbf{n}^{(2)}$.

4 Incoherent properties of the correlation tensor

Let us consider the incoherent mixture of factorizable two-particle spin states. In this case the two-particle density matrix is a sum of the direct products of one-particle density matrices only, with nonnegative coefficients:

$$\hat{\rho}^{(1,2)} = \sum_{\{s\}} \sum_{\{t\}} b_{\{s,t\}} \hat{\rho}_{\{s\}}^{(1)} \otimes \hat{\rho}_{\{t\}}^{(2)}, \quad b_{\{s,t\}} \geq 0, \quad \sum_{\{s\}} \sum_{\{t\}} b_{\{s,t\}} = 1. \tag{19}$$

Inserting the one-particle density matrices for spin–1/2 particles

$$\hat{\rho}_{\{s\}}^{(1)} = \frac{1}{2}(\hat{I}^{(1)} + \mathbf{P}_{\{s\}}^{(1)} \hat{\boldsymbol{\sigma}}^{(1)}), \quad \hat{\rho}_{\{t\}}^{(2)} = \frac{1}{2}(\hat{I}^{(2)} + \mathbf{P}_{\{t\}}^{(2)} \hat{\boldsymbol{\sigma}}^{(2)}) \tag{20}$$

into Eq. (19) and comparing with the general Eq. (3), one obtains the following expressions for the polarization vectors and the correlation tensor:

$$\mathbf{P}^{(1)} = \sum_{\{s\}} \sum_{\{t\}} b_{\{s,t\}} \mathbf{P}^{(1)}_{\{s\}}, \quad \mathbf{P}^{(2)} = \sum_{\{s\}} \sum_{\{t\}} b_{\{s,t\}} \mathbf{P}^{(2)}_{\{t\}},$$
$$T_{ik} = \sum_{\{s\}} \sum_{\{t\}} b_{\{s,t\}} P^{(1)}_{\{s\}i} P^{(2)}_{\{t\}k}. \tag{21}$$

Since the magnitudes of the polarization vectors do not exceed unity, $\mid \mathbf{P}^{(1)}_{\{s\}} \mid \leq 1$, $\mid \mathbf{P}^{(2)}_{\{t\}} \mid \leq 1$, it follows from Eq. (21) that, in the case of the incoherent mixture of the factorizable two-particle states, the diagonal components of the tensor T_{ik} satisfy the inequalities:

$$\mid T_{11} + T_{22} \mid \leq 1, \quad \mid T_{22} + T_{33} \mid \leq 1, \quad \mid T_{33} + T_{11} \mid \leq 1,$$
$$\mid \mathrm{tr} T \mid = \mid T_{11} + T_{22} + T_{33} \mid \leq 1. \tag{22}$$

It should be emphasized that the derived inequalities are related to the classical correlations at the probability (not amplitude) level. They simply reflect the fact that a weighted mean of the scalar products of some vectors cannot exceed the same mean of the corresponding products of vector modulae. In quantum mechanics, when we deal with the nonfactorizable coherent superpositions of two-particle states, the inequalities (22) may be substantially violated. Particularly, for a two–particle singlet state (see Eqs. (9)): $T_{11}+T_{22} = T_{11}+T_{33} = T_{22}+T_{33} = -2$, $\mathrm{tr}T = -3$. For a two–particle triplet state (see Eqs. (12) and (13)) the last of the inequalities (22) is always satisfied: $\mathrm{tr}T = 1$, while one of the other inequalities may be violated. Thus, $T_{11} + T_{22} \equiv 2w_0 > 1$ provided that $w_0 > 1/2$;[1] since then $2|\Re\rho_{+-}| < 1/2$, the remaining two inequalities are satisfied. On the other hand, for $w_0 < 1/2$, the violation of one of the inequalities $T_{11} + T_{33} = 1 - w_0 + 2\Re\rho_{+-} \leq 1$ or $T_{22} + T_{33} = 1 - w_0 - 2\Re\rho_{+-} \leq 1$ may happen; the required condition is $2|\Re\rho_{+-}| > w_0$.

5 Bell inequalities

The Bell inequalities [9] were obtained at the probability level in the framework of the concept of hidden parameters related to the common past of particles separated from each other in space during the detection; thus, the coherent properties of the quantum-mechanical superpositions of two-particle states were not taken into consideration. One of these inequalities, as applied to particles with spin 1/2, has the form [10]

$$Q = \mid \langle (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}) \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m}) \rangle + \langle (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}) \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m}') \rangle +$$
$$+ \langle (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}' \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m}) \rangle - \langle (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}') \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m}') \rangle \mid \leq 2, \tag{23}$$

where $\mathbf{n}$, $\mathbf{m}$, $\mathbf{n}'$ and $\mathbf{m}'$ are arbitrary unit vectors and

$$\langle (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}) \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m}) \rangle = \sum_i \sum_k T_{ik} n_i m_k \tag{24}$$

[1] Note that in case of a pure triplet state (a state described by the wave function), one can always achieve this condition (except for a completely polarized state $P = 1$) by a rotation which maximizes the probability of zero spin projection to a value $w_0^{\max} = (1 + \sqrt{1 - P^2})/2$.

is the average product of the double spin projections of the first and second particle onto different axes $\mathbf{n}$ and $\mathbf{m}$. The latter can be determined experimentally from the double distribution of the directions of the decay analyzers $\mathbf{n} = \mathbf{n}^{(1)}$ and $\mathbf{m} = \mathbf{n}^{(2)}$ using Eq.(14). Selecting the number of pairs of the decay analyzers $\Delta N(\mathbf{n}, \mathbf{m})$ in sufficiently narrow intervals of the solid angles $\Delta\Omega_{\mathbf{n}}$ and $\Delta\Omega_{\mathbf{m}}$, and denoting $W_\Delta(\mathbf{n}, \mathbf{m}) = \Delta N(\mathbf{n}, \mathbf{m})/[N\Delta\Omega_{\mathbf{n}}\Delta\Omega_{\mathbf{m}}]$, where N is the total number of pairs, one can calculate

$$\langle(\boldsymbol{\sigma}^{(1)}\mathbf{n}) \otimes (\boldsymbol{\sigma}^{(2)}\mathbf{m})\rangle = \frac{1}{\alpha_1\alpha_2}\left\{8\pi^2[W_\Delta(\mathbf{n}, \mathbf{m}) + W_\Delta(-\mathbf{n}, -\mathbf{m})] - 1\right\}. \tag{25}$$

It can be shown that the inequality (23) holds for the incoherent mixture of factorizable two-particle states with the density matrix defined according to Eqs. (19). Indeed, taking into account Eqs. (21) for the corresponding correlation tensor and the relations $\mathbf{m}+\mathbf{m}' = 2\mathbf{l}\cos(\beta/2)$, $\mathbf{m} - \mathbf{m}' = 2\mathbf{l}'\sin(\beta/2)$, where $\mathbf{l}$ and $\mathbf{l}'$ are the mutually perpendicular unit vectors, β is the angle between the vectors $\mathbf{m}$ and $\mathbf{m}'$, the quantity Q in Eq. (23) can be rewritten as

$$\begin{aligned} Q &=\mid \textstyle\sum_{\{s\}}\sum_{\{t\}} b_{\{s,t\}}[(\mathbf{P}^{(1)}_{\{s\}}\mathbf{n})(\mathbf{P}^{(2)}_{\{t\}}\mathbf{l})\, 2\cos(\frac{\beta}{2}) + (\mathbf{P}^{(1)}_{\{t\}}\mathbf{n}')(\mathbf{P}^{(2)}_{\{t\}}\mathbf{l}')\, 2\sin(\frac{\beta}{2})] \mid = \\ &=\mid \textstyle\sum_{\{s\}}\sum_{\{t\}} b_{\{s,t\}}P^{(2)(\|)}_{\{t\}}[(\mathbf{P}^{(1)}_{\{s\}}\mathbf{n})\, 2\cos(\frac{\beta}{2})\cos\alpha_{\{t\}} + (\mathbf{P}^{(1)}_{\{s\}}\mathbf{n}')\, 2\sin(\frac{\beta}{2})\sin\alpha_{\{t\}}] \mid . \end{aligned} \tag{26}$$

Here $P^{(2)(\|)}_{\{t\}}$ is the projection of the vector $\mathbf{P}^{(2)}_{\{t\}}$ onto the plane $(\mathbf{l}, \mathbf{l}')$, $\alpha_{\{t\}}$ is the angle between this projection and the vector $\mathbf{l}$. Since the magnitudes of the polarization vectors $\mathbf{P}^{(1)}_{\{s\}}$ and $\mathbf{P}^{(2)}_{\{t\}}$, as well as the magnitudes of their projections, cannot exceed unity, one finally proves the inequality (23):

$$Q \le \sum_{\{s\}}\sum_{\{t\}} 2b_{\{s,t\}} \max \mid \cos(\alpha_{\{t\}} \mp \frac{\beta}{2}) \mid \le 2\sum_{\{s\}}\sum_{\{t}\} b_{\{s,t\}} = 2. \tag{27}$$

Similar to the inequalities (22), the Bell inequality (23) can be violated for coherent superpositions of two–particle states. In particular, for the singlet state, in accordance with Eqs. (9) and (24) one has $\langle(\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}) \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m})\rangle = -\mathbf{nm}$, so that the quantity

$$Q =\mid -\mathbf{nm} - \mathbf{nm}' - \mathbf{n}'\mathbf{m} + \mathbf{n}'\mathbf{m}' \mid . \tag{28}$$

As a result, the maximal possible violation of the Bell inequality: $Q_{max} = 2\sqrt{2} > 2$ corresponds to the situation when the unit vectors are selected in the same plane and satisfy the conditions $\mathbf{n} \perp \mathbf{n}'$, $\mathbf{m} \perp \mathbf{m}'$, $\mathbf{nm} = \mathbf{nm}' = \mathbf{n}'\mathbf{m} = \cos(\pi/4) = 1/\sqrt{2}$, $\mathbf{n}'\mathbf{m}' = \cos(3\pi/4) = -1/\sqrt{2}$.

The analogous violations of the Bell inequality take place also for the triplet state with the zero projection onto any axis z, provided the vectors $\mathbf{n}$, $\mathbf{m}$, $\mathbf{n}'$ and $\mathbf{m}'$ are selected in the plane (x, y). For such vectors (see Eq. (11) with $w_0 = 1$) $\langle(\boldsymbol{\sigma}^{(1)}\mathbf{n}) \otimes (\boldsymbol{\sigma}^{(2)}\mathbf{m})\rangle = +\mathbf{nm}$, so that the quantity Q coincides with the singlet expression in Eq. (28). In the general aligned triplet state (a state with the diagonal density matrix), the double spin correlation $\langle(\boldsymbol{\sigma}^{(1)}\mathbf{n}) \otimes (\boldsymbol{\sigma}^{(2)}\mathbf{m})\rangle$ in the plane (x, y) and the corresponding quantity Q scale with the probability w_0 of the zero spin projection onto the z–axis. The Bell inequality (23) can then be violated for $w_0 > 1/\sqrt{2}$ only. Recall that the inequality $|T_{11} + T_{22}| \le 1$ appears to be more stringent, being always violated for $w_0 > 1/2$.

It should be stressed that there exists the difference of principle between the singlet state in quantum mechanics and the incoherent mixture of the products of two one-particle states with opposite projections onto the isotropically distributed axes. In the latter case $T_{ik} = -\delta_{ik}/3$, $\langle(\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{n}) \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{m})\rangle = -\mathbf{nm}/3$, so that the inequalities (22), as well as the Bell inequality (23), are valid.

6 Spin correlations

Consider, for example, the processes $e^-e^+, q\bar{q} \to \mu^-\mu^+$ well below the Z^0 threshold or the process $q\bar{q} \to t\bar{t}$. Due to parity conservation, the corresponding dominant tree diagrams (the s-channel photon or gluon exchange) select the final particles in a triplet state, i.e. in a state with correlated particle spins.[2] Directing the z-axis parallel to the production plane normal and the y-axis - antiparallel to the muon (top-quark) c.m.s. velocity vector $\boldsymbol{\beta}$, the nonzero components of the correlation tensor in the tree approximation become (see, e.g., [16]):

$$T_{11} = A^{-1}(2-\beta^2)\sin^2\theta, \quad T_{22} = A^{-1}(2\cos^2\theta + \beta^2\sin^2\theta), \quad T_{33} = -A^{-1}\beta^2\sin^2\theta,$$
$$T_{12} = T_{21} = -A^{-1}\gamma^{-1}\sin 2\theta, \quad A = 2 - \beta^2\sin^2\theta, \tag{29}$$

where θ, β and $\gamma = (1-\beta^2)^{-1/2}$ are the c.m.s. muon (top-quark) production angle, velocity and Lorentz factor. Using the transformation properties of the correlation tensor under the rotations of the coordinate system, one can easily prove that the choice of the quantization axis parallel to the production plane normal maximizes the probability of zero spin projection: $w_0^{\max} = (2-\beta^2\sin^2\theta)^{-1}$. Since for a triplet state $T_{11} + T_{22} = 2w_0$, the first of the inequalities (22) will be violated provided $\beta\sin\theta \neq 0$.

It should be noted that near threshold, the considered processes are strongly influenced by the Coulomb or colour-Coulomb final state interaction (FSI). Taking into account the point-like character of these processes and neglecting the effect of finite lifetimes of produced particles (see, however, [17]) the FSI can be approximately taken into account by multiplying the production amplitudes by the stationary solutions of the Coulomb scattering problem at zero separation: $\psi_{\mathbf{k}^*}^{c(-)}(0) = e^{-i\delta_c}A_c^{1/2}$, where $\mathbf{k}^*$ is the c.m.s. muon (top-quark) 3-momentum, $\delta_c = \arg\Gamma(1+i\eta)$ is the Coulomb s-wave shift, $A_c = 2\pi\eta/[\exp(2\pi\eta)-1]$ is the Coulomb penetration factor, $\eta = (ak^*)^{-1}$, $a = (-\alpha\mu)^{-1}$ is the Bohr radius (taken negative in case of attraction), μ is the pair reduced mass and α is here the fine structure constant for $\mu^-\mu^+$-pair while $\alpha = \frac{4}{3}\alpha_s$ $(-\frac{1}{6}\alpha_s)$ for colour singlet (octet) $t\bar{t}$-pair [15]. It is important that this interaction, being spin-independent, has no influence on the spin structure of the amplitudes of the considered processes. Particularly, it leaves unchanged the components of the correlation tensor in Eqs. (29).

For identical particles, independent of the production dynamics, the effect of Bose or Fermi quantum statistics (QS) leads to the spin correlation at small relative momenta

[2] The processes $q\bar{q} \to \mu^-\mu^+, t\bar{t}$ are relevant for the production of Drell-Yan dimuons and top-quark pairs in hadronic collisions. The contribution of the competing process $gg \to t\bar{t}$ is small at Tevatron, it will however dominate at LHC. Note that the spin composition of the $t\bar{t}$-pair in this process changes with the increasing energy from the singlet (due to the Landau-Yang theorem forbidding the total two–gluon angular momentum $J = 1$) [15] to the triplet (due to the helicity conserving gluon coupling to the relativistic quarks).

$Q = 2k^*$ in the two–particle rest frame. This is obvious due to the fact that the total spin S of two identical particles and the orbital angular momentum L in their c.m.s. satisfy the well-known equality [18]: $(-1)^{S+L} = 1$. When the momentum difference Q approaches zero, the states with nonzero orbital angular momenta disappear, and only those with $L = 0$ and even total spin S remain. As a result, at $Q \to 0$, two identical spin–1/2 particles (e.g., two protons or two Λ-particles) can be produced only in the singlet state [6]. This conclusion is clearly model independent. The corresponding width of the singlet enhancement or triplet suppression is inversely related to the effective radius r_0 of the emission region [12, 13]. The measurement of the singlet or triplet fractions ρ_s or ρ_t thus yields similar though completely independent information on the space–time separation of the produced particles as the standard correlation femtoscopy technique. The latter exploits the fact that the correlation function at small Q is sensitive to particle separation due to the effects of QS, Coulomb and strong FSI. Note that the correlation function can be defined as a ratio $R(p_1, p_2)$ of the two–particle production cross section to the reference one which would be observed in the absence of the effects of QS and FSI. The reference distribution is usually constructed by mixing the particles from different events.

In the model of independent one–particle sources, the effects of QS and FSI are taken into account merely multiplying the initial amplitude for a total spin S by a properly symmetrized stationary solution of the scattering problem $\psi^{S(-)}_{\mathbf{k}^*}(\mathbf{r}^*) = [\psi^{S(+)}_{-\mathbf{k}^*}(\mathbf{r}^*)]^* \equiv \psi^{S*}$, where $\mathbf{k}^* = \mathbf{p}_1^* = -\mathbf{p}_2^*$ and $\mathbf{r}^* = \mathbf{r}_1^* - \mathbf{r}_2^*$ (the minus sign of the vector $\mathbf{k}^*$ in $\psi^{S(+)}_{-\mathbf{k}^*}(\mathbf{r}^*)$ corresponds to the reverse in time direction of the emission process). Particularly, one has [19, 20]: $R(p_1, p_2) = \sum_S \widetilde{\rho}_S \langle|\psi^{S(+)}_{-\mathbf{k}^*}(\mathbf{r}^*)|^2\rangle_S \equiv \sum_S R_S(p_1, p_2)$ and $\rho_S = \widetilde{\rho}_S\langle|\psi^S|^2\rangle/R \equiv R_S/R$. The averaging is done over the emission points of the two particles in a state with total spin S populated in the absence of QS and FSI with the probability $\widetilde{\rho}_S$, $\sum_S \widetilde{\rho}_S = 1$. For spin–1/2 particles initially emitted independently with the polarizations $\widetilde{\mathbf{P}}_1$ and $\widetilde{\mathbf{P}}_2$: $\widetilde{\rho}_s = (1 - \widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{P}}_2)/4$, $\widetilde{\rho}_t = (3 + \widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{P}}_2)/4$. The expressions for the components of the correlation tensor and particle polarizations are rather lengthy and can be found in ref. [13]. Here we present only the simple result for the case of low–Q pairs of identical spin–1/2 particles when one can put $\widetilde{\mathbf{P}}_1 \doteq \widetilde{\mathbf{P}}_2 \doteq \widetilde{\mathbf{P}}$. Then

$$\mathbf{P} = \widetilde{\mathbf{P}}\rho_t/\widetilde{\rho}_t, \quad T_{ik} = \widetilde{P}_i\widetilde{P}_k\rho_t/\widetilde{\rho}_t + \delta_{ik}\left(\rho_t/\widetilde{\rho}_t - 1\right). \tag{30}$$

It may be seen from Eqs. (30) that, in the presence of QS and FSI, the spins of the initially unpolarized ($\widetilde{P}_{1i} = \widetilde{P}_{2i} = 0$) and uncorrelated ($\widetilde{T}_{ik} = 0$) particles remain unpolarized but not uncorrelated: $T_{ik} = -\delta_{ik}\langle|\psi^s|^2 - |\psi^t|^2\rangle/\langle|\psi^s|^2 + 3|\psi^t|^2\rangle$. For $\psi^t = 0$ ($\psi^s = 0$) the latter tensor reduces to a pure singlet (triplet) one: $T^s_{ik} = -\delta_{ik}$ ($T^t_{ik} = \frac{1}{3}\delta_{ik}$). On the other hand, for initially polarized identical particles the polarization vectors vanish and $T_{ik} \to T^s_{ik}$ at $Q \to 0$ due to forbidden triplet amplitude ψ^t at $Q = 0$.

It should be emphasized that the correlation of the polarizations of two spin–1/2 particles, conditioned by their identity, is maximal for $Q \to 0$ (tr$T \to -3$ independently of the value of the initial polarization $\widetilde{\mathbf{P}}$ - the singlet state) . At sufficiently large Q, the model of independent one-particle sources yields $T_{ik} = \widetilde{P}_i\widetilde{P}_k$ so that the spin correlations vanish for unpolarized particles (tr$T \to 0$).

Note that recent measurements of $\Lambda\Lambda$ correlations in multihadronic Z^0 decays at LEP point to rather small effective radius of the Λ emission region: $r_0 \sim 0.1 - 0.2$ fm [21]. One can therefore expect that at $Q < 1/r_0 \sim 1 - 2$ GeV/c the system of two Λ-

hyperons is created mainly in the singlet state.[3] For such systems, the inequalities (22) for the sums of the components of the correlation tensor and the Bell inequality (23) could then be violated. Indeed, the expected suppression of the triplet fraction at small Q was observed in several LEP experiments [21, 22]. Particularly, the ALEPH result: $\rho_t = 0.36 \pm 0.30 \pm 0.08$ at $Q = 0 - 1.5$ GeV/c indicates the violation of the inequality $|\mathrm{tr}T| \equiv |4\rho_t - 3| \leq 1$.

7 Conclusions

We have performed the theoretical analysis of spin correlations in the system of two spin–1/2 particles using their asymmetric (parity violating) decays. It is shown that the spin correlation tensor can be determined from the angular correlations of the decay analyzers, particularly, for two Λ–particles both decaying into the channel $\Lambda \rightarrow p\pi^-$, - from the correlations of the directions of the decay protons in the respective Λ–rest frames.

We have derived the inequalities, including those of Bell, for linear combinations of the components of the spin correlation tensor valid in the case of incoherent mixture of two-particle factorizable spin states. The violation of these inequalities is connected with the general quantum-mechanical effect (first considered by Einstein, Podolsky and Rosen) and can serve as a crucial test of the basic principles of quantum mechanics.

We have considered some examples of the processes allowing one to verify the consequences of the quantum–mechanical coherence with the help of the two–particle spin correlations measured in asymmetric particle decays, the coherence arising either due to the production dynamics (dominant triplet states in the processes $e^-e^+, q\bar{q} \rightarrow \mu^-\mu^+, t\bar{t}$) or due to the effect of quantum statistics at small relative momenta (dominant singlet $\Lambda\Lambda$-state).

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References

[1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47** (1935) 477.

[2] D.R. Inglis, Rev. Mod. Phys. **33** (1961) 1.

[3] T.B. Day, Phys. Rev. **121** (1961) 1204.

[4] V.I.Ogievetsky, M.I.Podgoretsky, Zh. Exp. Teor. Phys. **43** (1962) 1362.

[5] V.L. Lyuboshitz, E.O. Okonov, Sov. J. Nucl. Phys. **4** (1967) 859.

[6] V.L. Lyuboshitz, M.I. Podgoretsky, Phys. At. Nucl. **59** (1996) 449.

[3] A large statistics of $\Lambda\Lambda$ pairs is also available in heavy ion collisions at SPS and soon will be accumulated at RHIC. However, due to much larger effective radius r_0 of several fm, the singlet state dominates here only in quite narrow region of $Q < 0.1$ GeV/c.

[7] V.L. Lyuboshitz, M.I. Podgoretsky, Phys. At. Nucl. **60** (1997) 39;
V.L. Lyuboshitz, JINR **E2-98-274**, Dubna (1998); Proc. XIV ISHEPP, Relativistic Nuclear Physics and Quantum Chromodynamics, vol. II , p. 143, Dubna, 2000.

[8] V.L. Lyuboshitz, V.V. Lyuboshitz, Phys. At. Nucl. **63** (2000) 767.

[9] S. Bell, Rev. Mod. Phys. **38** (1965) 447; Physica **1** (1965) 1951.

[10] A.A. Grib, Usp. Fiz. Nauk **142** (1984) 619.

[11] L. Wolfenstein, Phys. Rev. **75** (1949) 1664;
L. Wolfenstein, J. Ashkin, Phys. Rev. **85** (1952) 947.

[12] G. Alexander, H.J. Lipkin, Phys. Lett. **B352** (1995) 162.

[13] R. Lednicky, MPI-PhE/99-10, Munich (1999).

[14] G. Mahlon, S. Parke, Phys. Rev. **D53** (1996) 4886.

[15] Y. Hara, Prog. Theor. Phys. **86** (1991) 779.

[16] A. Brandenburg, Phys. Lett. **B388** (1996) 626.

[17] V.S. Fadin, V.A. Khoze, Sov. J. Nucl. Phys. **48** (1988) 309; ibid. **53** (1991) 692.

[18] L.D. Landau, E.M. Lifshitz: Quantum Mechanics: Nonrelativistic Theory, Sect. 62 (London, Pergamon, 1965).

[19] R. Lednicky, V.L. Lyuboshitz: Sov. J. Nucl. Phys. **35** (1982) 770; Proc. CORINNE 90, Nantes, France, 1990 (ed. D. Ardouin, World Scientific, 1990) p. 42.

[20] V.L. Lyuboshitz, Sov. J. Nucl. Phys. **48** (1988) 956.

[21] ALEPH Collab., Phys. Lett. **B475** (2000) 395.

[22] OPAL Collab., Phys. Lett. **B384** (1996) 377;
DELPHI Collab.: Proc. ICHEP'98, Vancouver 1998, Ref. 154.